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Finite-amplitude effects on steady lee-wave patterns in subcritical stratified flow over topography

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The flow of a continuously stratified fluid over a smooth bottom bump in a channel of finite depth is considered. In the weakly nonlinear–weakly dispersive régime $\epsilon = a/h \ll 1$, $\mu = h/l \ll 1$ (where h is the channel depth and a, l are the peak amplitude and the width of the obstacle respectively), the parameter $A = \epsilon/\mu^p$ (where $p > 0$ depends on the obstacle shape) controls the effect of nonlinearity on the steady lee wavetrain that forms downstream of the obstacle for subcritical flow speeds. For $A = O(1)$, when nonlinear and dispersive effects are equally important, the interaction of the long-wave disturbance over the obstacle with the lee wave is fully nonlinear, and techniques of asymptotics 'beyond all orders' are used to determine the (exponentially small as $\mu \to 0$) lee-wave amplitude. Comparison with numerical results indicates that the asymptotic theory often remains reasonably accurate even for moderately small values of μ and ϵ, in which case the (formally exponentially small) lee-wave amplitude is greatly enhanced by nonlinearity and can be quite substantial. Moreover, these findings reveal that the range of validity of the classical linear lee-wave theory ($A \ll 1$) is rather limited.

1. Introduction

When a stably stratified fluid flows over topography, the internal gravity waves that form downstream are commonly referred to as lee waves; in many respects, they are akin to the more familiar gravity waves induced on the free surface of a homogeneous fluid by a moving disturbance. Lee waves are of fundamental meteorological interest, as they are often present on the lee side of mountains owing to the prevailing winds and are believed to play an important part in the development of storms (see, for example, Lilly 1978).

We shall focus on the generation of lee-wave disturbances in the simple setting of continuously stratified flow over a two-dimensional bottom obstacle in a channel bounded by rigid walls. As in the analogous problem of gravity waves on the free surface of a homogeneous fluid layer, it proves useful to introduce the long-wave parameter $\mu = h/l$ and the nonlinearity parameter $\epsilon = a/h$, where h is the channel depth and a, l are, respectively, the peak amplitude and the characteristic lengthscale of the obstacle in the streamwise direction.

These two parameters along with the Froude number $F = U/(h\mathcal{N}_0)$, U being the undisturbed flow speed and $\mathcal{N}_0$ a characteristic value of the Brunt–Väisälä frequency, define various flow régimes. Specifically, the linear lee-wave régime is obtained when

$\epsilon \to 0$, $\mu = O(1)$ and has been analysed extensively in previous work using the linearized equations of motion (see Miles 1969 for a review). In this limit, the induced steady lee-wave pattern comprises a finite number of internal-wave modes. The wavenumber of each of these modes is such that the phase speed matches the flow speed, and, for this to be possible, it follows from the linear dispersion relation (see, for example, Yih 1979, Ch. 5, §4) that the Froude number has to be subcritical relative to each mode that is excited – the flow speed has to be less than the corresponding long-wave speed. The lee-wave amplitude depends on the specific obstacle shape and may be found by Fourier-transform techniques. In particular, in the hydrostatic limit (when the width of the obstacle is large compared to the channel depth, $\mu \ll 1$), dispersive effects are weak and the lee-wave amplitude is exponentially small with respect to μ.

The flow characteristics in the nonlinear régime are more difficult to analyse quantitatively, however, because the governing equations are nonlinear in general when ϵ is finite, and previous efforts centre on two approaches that are valid under particular conditions. Specifically, Long's model is based on the observation that the Euler equations of two-dimensional steady flow become linear for certain density and velocity profiles of the background flow, assuming that the flow remains undisturbed far upstream (Dubreil-Jacotin 1935; Long 1953; Yih 1960). In the case of constant upstream flow speed which is of interest here, these conditions are met for a weakly stratified (Boussinesq) fluid with uniform stratification (constant Brunt–Väisälä frequency). Long's model has been used primarily to obtain theoretical estimates of the critical obstacle steepness (keeping the other flow parameters fixed) above which density inversions occur and the assumption of steady flow is expected to fail owing to static instability (Miles 1969; Miles & Huppert 1969). For finite-amplitude obstacles, the validity of Long's hypothesis of no upstream influence has been questioned (Baines 1977), however, and the issue has not been settled completely as yet (see, for example, the recent numerical study by Lamb 1994).

The second approach has received attention more recently and is valid in the weakly nonlinear régime ($0 < \epsilon \ll 1$) near resonance – when the flow speed is close to the long-wave speed of an internal-wave mode in the channel. Under these conditions, the response is dominated by the resonant mode and the evolution of the corresponding amplitude is governed by a forced Korteweg–de Vries (fKdV) equation (Grimshaw & Smyth 1986). For transcritical flow speeds, strong upstream influence is found in the form of solitary waves that are generated periodically, as in the analogous problem of free-surface flow of a homogeneous fluid near resonance (Akylas 1984; Cole 1985). In the special case of a uniformly stratified Boussinesq fluid, when Long's model applies, the fKdV equation is replaced by an integral–differential evolution equation which reveals that density inversions often appear during the transient development of the response (Grimshaw & Yi 1991).

We shall concentrate on finite-amplitude lee-wave disturbances for subcritical flow speeds (away from resonant conditions). Our approach is complementary to the works cited above: as in Long's model, attention is confined to steady waves under the assumption of no upstream influence, but the flow may have general (stable) stratification. Moreover, while the asymptotic theory developed here is formally valid in the weakly nonlinear–weakly dispersive régime ($\epsilon, \mu \ll 1$), comparison with numerical solutions indicates that the analytical results often are reasonably accurate for a wide range of moderately small values of ϵ and μ – sometimes even close to conditions where density inversions are about to occur.

The asymptotic theory is motivated by an earlier study, based on the (steady) fKdV equation, of short-scale wavetrains generated by steady forcings (Akylas & Yang 1995). This model problem suggests that nonlinear effects can significantly modify the lee-wave disturbance induced by a long obstacle ($\mu \ll 1$), even though the wave amplitude is exponentially small with respect to μ. More precisely, in the joint limit $\epsilon, \mu \to 0$, it turns out that the relative importance of finite-amplitude effects is measured by the parameter

$$A = \frac{\epsilon}{\mu^p}, \tag{1.1}$$

where $p > 0$ depends on the obstacle shape. In particular, the classical linear lee-wave theory is expected to be valid when $A \ll 1$, so its usefulness is limited – the situation is reminiscent of the 'long-wave paradox' in weakly nonlinear shallow-water waves (Ursell 1953).

For $A = O(1)$, when nonlinear and dispersive effects are equally important, the interaction of the long-wave disturbance over the obstacle with the relatively short lee wave downstream is, in fact, fully nonlinear. As a result, it is necessary to account for all nonlinear and dispersive terms in the governing equations in order to determine the lee-wave amplitude; following Akylas & Yang (1995), this is carried out using techniques of asymptotics 'beyond all orders' (see, for example, Segur, Tanveer & Levine 1991).

The predictions of the asymptotic theory are compared to numerically computed wave patterns in subcritical stratified flow over two possible obstacle shapes and when the square of the Brunt–Väisälä frequency is either constant or varies linearly with depth. As expected, there is good agreement in the limit $\epsilon, \mu \to 0$, confirming the validity of the asymptotic theory. More interestingly, however, the asymptotic results often remain reasonably accurate for moderately small values of ϵ and μ; in this range, the (formally exponentially small) lee-wave amplitude can be quite substantial – in some cases several times larger than the estimate obtained from linear theory – owing to the effect of nonlinearity.

In a recent numerical study of two-layer flow over topography, Belward & Forbes (1993) also report that the lee-wave amplitude is enhanced owing to nonlinear effects. It would appear that the approach taken here can be extended to discuss layered flows.

2. Formulation and linear theory

Consider two-dimensional steady flow of an inviscid, incompressible, density-stratified fluid along a channel that is bounded by rigid walls and features a locally confined bottom bump.

We shall use dimensionless variables taking the uniform channel depth h (away from the bottom bump) as the vertical lengthscale, the characteristic width l of the bump as the horizontal lengthscale, and $\mathcal{N}_0^{-1}$, the inverse of a characteristic value of the Brunt–Väisälä frequency, as the timescale. Far upstream ($x \to -\infty$), the flow velocity is assumed to be uniform, and the density $\rho_0(z)$ varies with height z in a prescribed (stable) way. The Brunt–Väisälä frequency $\mathcal{N}(z)$ is then given by

$$\beta \rho_0 \mathcal{N}^2 = -\rho_{0z},$$

where $\beta = \mathcal{N}_0^2 h/g$ is the Boussinesq parameter, g being the gravitational acceleration.

For incompressible flow, it is convenient to work with the stream function $\Psi = Fz + F\psi$, where $\psi(x,z)$ describes the disturbance induced by the bottom obstacle and

F is the Froude number, $F = U/(h\mathcal{N}_0)$, U being the upstream flow speed. In terms of Ψ, the horizontal and vertical velocity components are given by Ψ_z, $-\mu\Psi_x$ $(\mu = h/l)$ respectively; thus, incompressibility is automatically satisfied.

Assuming further that all streamlines originate far upstream where the flow remains undisturbed (no upstream influence),

$$\psi \to 0 \qquad (x \to -\infty), \tag{2.1}$$

the momentum equations and the equation of mass conservation can be combined, following the procedure outlined in Akylas & Grimshaw (1992), into a single equation for ψ:

$$\mu^2\psi_{xx} + \psi_{zz} + \mathcal{N}^2(z+\psi)\left\{\frac{\psi}{F^2} - \beta\psi_z - \tfrac{1}{2}\beta\left(\mu^2\psi_x^2 + \psi_z^2\right)\right\} = 0. \tag{2.2}$$

This equation, which is usually referred to as Long's equation (Long 1953), is valid under the assumptions stated above within the channel $-\infty < x < \infty$, $\epsilon f(x) \leqslant z \leqslant 1$. Here $z = \epsilon f(x)$ defines the bottom of the channel where the bump $(f(x) \to 0,\ x \to \pm\infty)$ is present, and $\epsilon = a/h$, a being the peak height of the bump.

Finally, the appropriate boundary conditions, which ensure that the channel walls are streamlines, are

$$\psi = -\epsilon f(x) \qquad (z = \epsilon f(x)), \tag{2.3a}$$

$$\psi = 0 \qquad (z = 1). \tag{2.3b}$$

For a finite-amplitude obstacle, the problem posed by (2.1)–(2.3) is nonlinear in general. Before turning our attention to nonlinear effects, however, we shall briefly review the salient features of the linear response. In the small-amplitude limit $(\epsilon \to 0)$, $\psi = O(\epsilon)$ so the governing equation (2.2) and the bottom boundary condition (2.3a) may be linearized. The resulting linear boundary-value problem then can be readily solved by Fourier transform,

$$\psi(x,z) = \int_{\mathscr{C}^-} \mathrm{e}^{\mathrm{i}kx}\widehat{\psi}(k,z)\mathrm{d}k, \tag{2.4}$$

indenting the contour $\mathscr{C}^-$ to pass below all singularities of $\widehat{\psi}$ on the real k-axis so that (2.1) is met. It turns out that

$$\widehat{\psi}(k,z) = -\epsilon\widehat{f}(k)\left\{q(z) + \mu^2k^2\sum_{r=1}^{\infty}\frac{\gamma_r}{\kappa_r - \mu^2k^2}\,\phi_r(z)\right\}, \tag{2.5}$$

where $\widehat{f}(k)$ is the Fourier transform of the obstacle shape $f(x)$,

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}f(x)\mathrm{d}x.$$

The first term on the right-hand side of (2.5) corresponds to the purely hydrostatic $(\mu = 0)$ linear response and is given in terms of the solution to the boundary-value problem

$$(\rho_0 q_z)_z + \frac{\rho_0}{F^2}\mathcal{N}^2 q = 0 \qquad (0 \leqslant z \leqslant 1), \tag{2.6a}$$

$$q(0) = 1, \qquad q(1) = 0. \tag{2.6b}$$

The term including the infinite sum in expression (2.5) accounts for dispersive corrections to the hydrostatic response owing to the internal-wave modes $\{\phi_r(z), \kappa_r\}$ $(r =$

$1,2,\ldots$). This orthogonal and complete set of modes is defined by the eigenvalue problem

$$(\rho_0\phi_{rz})_z + \rho_0\left\{\frac{\mathcal{N}^2}{F^2} - \kappa_r\right\}\phi_r = 0 \qquad (0 \leqslant z \leqslant 1), \tag{2.7a}$$

$$\phi_r = 0 \qquad (z = 0, 1), \tag{2.7b}$$

and satisfy the orthogonality relation

$$\int_0^1 \rho_0\phi_r\phi_s \mathrm{d}z = \delta_{rs}, \tag{2.8}$$

δ_{rs} being the Kronecker delta. In particular, the constants γ_r in (2.5) are the participation factors of these modes:

$$\gamma_r = \int_0^1 \rho_0 q\phi_r \mathrm{d}z.$$

It is clear from (2.5) that $\widehat{\psi}$ has simple-pole singularities on the real k-axis at $k = \pm\kappa_r^{1/2}/\mu \equiv \pm k_r$ when $\kappa_r > 0$ for some r, and, upon evaluating the contour integral in (2.4), these singularities contribute lee waves downstream ($x > 0$). The sign of κ_r in turn depends on whether the Froude number F is subcritical or supercritical relative to c_r, the corresponding linear long-wave speed. These are the speeds of the linear long-wave modes $\{f_s(z), c_s\}$ ($s = 1, 2, \ldots$) (with $c_1 > c_2 > \cdots > 0$) defined by the eigenvalue problem

$$(\rho_0 f_{sz})_z + \frac{\rho_0}{c_s^2}\mathcal{N}^2 f_s = 0 \qquad (0 \leqslant z \leqslant 1), \tag{2.9a}$$

$$f_s = 0 \qquad (z = 0, 1). \tag{2.9b}$$

If now $c_{N+1} < F \leqslant c_N$ for some $N \geqslant 1$, it follows that $\kappa_1 > \kappa_2 > \cdots > \kappa_N \geqslant 0 > \kappa_{N+1} > \cdots$ and, in particular, $\kappa_N = 0$ for $F = c_N$ (Yih 1979, Ch. 5, §4.1.3). Hence, lee waves are excited only when F is subcritical relative to at least c_1, the highest of the linear long-wave speeds. This is easy to understand physically because the phase speed of each internal-wave mode is known to decrease with wavenumber (Yih 1979, Ch. 5, §4.1.4), so stationary (in the frame of the obstacle) lee waves are not possible if the flow speed is supercritical relative to all linear long-wave speeds.

We also remark here that the steady linear response is singular when the flow speed is critical with respect to a long-wave speed – the inhomogeneous problem (2.6*b*) has no solution since, from (2.9*b*), the long-wave mode $f_N(z)$ is a non-trivial homogeneous solution when $F = c_N$ ($\kappa_N = 0$). This is the resonant case where, as mentioned earlier, the assumptions of steady flow and no upstream influence fail in a finite range of transcritical flow speeds (Grimshaw & Smyth 1986; Grimshaw & Yi 1991). As our interest centres on steady lee-wave disturbances, we shall take the flow speed to be subcritical relative to at least one (and not close to any) linear long-wave speed, viz. $c_{N+1} < F < c_N$ with $N \geqslant 1$.

Returning then to (2.4) and (2.5), we conclude that the downstream response consists of a finite number of lee waves:

$$\psi \sim -2\pi\epsilon\sum_{r=1}^{N}\gamma_r k_r \widehat{f}(k_r)\phi_r(z)\sin k_r x \qquad (x \to \infty),$$

taking the obstacle to be symmetric, $f(x) = f(-x)$. If, furthermore, the obstacle is long relative to the channel depth ($\mu \ll 1$), the induced lee waves have short wavelength relative to the obstacle length (the lee wavelength is comparable to the channel depth) and exponentially small amplitude (the Fourier transform of a smooth obstacle profile $f(x)$ is exponentially small for large wavenumbers, $k_r \gg 1$). Accordingly, in this nearly hydrostatic régime, the lee wave of mode N which has the relatively longer wavelength ($k_N < k_{N-1} < \cdots < k_1$) dominates:

$$\psi \sim -2\pi\epsilon\gamma_N k_N \widehat{f}(k_N)\phi_N(z)\sin k_N x \qquad (x \to \infty;\ \mu \ll 1). \tag{2.10}$$

However, we shall demonstrate in the ensuing discussion that the validity of this expression for the induced lee-wave disturbance is severely limited by finite-amplitude effects. Specifically, it follows from the asymptotic theory developed below that, close to the hydrostatic limit ($\mu \ll 1$), the relative importance of nonlinearity is controlled by the parameter A defined in (1.1); the linear, nearly hydrostatic approximation (2.10) is valid only when $\epsilon \ll \mu^p$, where p depends on the particular obstacle shape.

3. Asymptotic theory

Attention is now focused on steady lee-wave disturbances in the weakly nonlinear–weakly dispersive régime ($\epsilon, \mu \ll 1$). As indicated by the linear solution sketched in §2, lee waves in the physical domain go hand in hand with simple-pole singularities of $\widehat{\psi}(k,z)$ on the real k-axis in the wavenumber domain, and the lee-wave amplitude is completely determined by the corresponding residues. Accordingly, following the perturbation procedure devised by Akylas & Yang (1995), we shall compute the residues of $\widehat{\psi}(k,z)$ at the dominant lee wavenumber $k = \pm k_N$ asymptotically for $\epsilon, \mu \ll 1$.

To this end, returning to the full nonlinear governing equation (2.2) and taking the Fourier transform formally, it is found that $\widehat{\psi}$ satisfies

$$\widehat{\psi}_{zz} - \beta\mathcal{N}^2\widehat{\psi}_z + \left(\frac{\mathcal{N}^2}{F^2} - \mu^2k^2\right)\widehat{\psi} - \tfrac{1}{2}\beta\mathcal{N}^2\left(\widehat{\psi_z^2} + \mu^2\widehat{\psi_x^2}\right)$$
$$+\sum_{j=1}^{\infty} M_j\left\{\frac{\widehat{\psi^{j+1}}}{F^2} - \beta\widehat{\psi^j\psi_z} - \tfrac{1}{2}\beta\left(\widehat{\psi^j\psi_z^2} + \mu^2\widehat{\psi^j\psi_x^2}\right)\right\} = 0, \tag{3.1}$$

where
$$\mathcal{N}^2(z+\psi) = \mathcal{N}^2(z) + \sum_{j=1}^{\infty} M_j(z)\psi^j$$

with
$$M_j \equiv \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}z^j}\mathcal{N}^2(z).$$

Similarly, the boundary conditions (2.3) (after expanding (2.3a) about $z = 0$) transform to

$$\widehat{\psi} + \sum_{j=1}^{\infty}\frac{\epsilon^j}{j!}\widehat{\frac{\partial^j\psi}{\partial z^j}f^j} = -\epsilon\widehat{f} \qquad (z = 0), \tag{3.2a}$$

$$\widehat{\psi} = 0 \qquad (z = 1). \tag{3.2b}$$

Expanding then $\widehat{\psi}$ in powers of ϵ and μ^2, it follows from (3.1), (3.2) that

$$\widehat{\psi}(k,z) = -\epsilon \widehat{f}(k)\left\{ q(z) + \mu^2 k^2 \sum_{r=1}^{\infty} \frac{\gamma_r}{\kappa_r}\,\phi_r(z) + \cdots \right\}$$
$$+\,\epsilon^2 \widehat{f^2}(k)\left\{ q'(0)q(z) + \sum_{r=1}^{\infty} \frac{\delta_r}{\kappa_r}\,\phi_r(z) \right\} + \cdots, \tag{3.3}$$

where
$$\delta_r = -\int_0^1 \rho_0 \phi_r \left(\frac{M_1}{F^2} q^2 - \beta M_1 q q' - \tfrac{1}{2}\beta \mathcal{N}^2 q'^2 \right) \mathrm{d}z.$$

The $O(\epsilon)$ term in this expansion corresponds to the linear hydrostatic solution; the rest of the terms are the leading-order nonlinear and dispersive corrections.

In view of the poles at $k = \pm k_r$ of the linear dispersive solution (2.5), it is expected that expansion (3.3) will become disordered when $k = O(1/\mu)$ owing to dispersive effects. There is a second non-uniformity in (3.3), however, that derives from nonlinear effects: since $f^2(x)$ is generally steeper than $f(x)$ in the physical domain, $\widehat{f^2}(k)$ goes to zero less rapidly than $\widehat{f}(k)$ as $|k| \to \infty$ in the wavenumber domain.

To be specific, for the obstacle profile $f(x) = \operatorname{sech}^p x$, where p is a positive integer,

$$\widehat{f}(k) = O\left(|k|^{p-1} \mathrm{e}^{-\pi|k|/2}\right), \quad \widehat{f^2}(k) = O\left(|k|^{2p-1} \mathrm{e}^{-\pi|k|/2}\right) \qquad (|k| \to \infty);$$

hence
$$\frac{\widehat{f^2}(k)}{\widehat{f}(k)} = O(|k|^p) \qquad (|k| \to \infty),$$

resulting in a disordering of (3.3) when $k = O(\epsilon^{-1/p})$. For this class of obstacles, therefore, it is now clear that the classical linear lee-wave theory is valid – nonlinearity does not affect the residues of the poles of $\widehat{\psi}(k,z)$ at $k = \pm k_N$ – only when $\epsilon \ll \mu^p$; in terms of the parameter $A = \epsilon/\mu^p$, this condition is met when $A \ll 1$. Note, however, that p is also the order of the pole singularities (closest to the real axis) of the obstacle shape $f(x) = \operatorname{sech}^p x$ in the complex plane; with this interpretation of p, the parameter A can be defined for any smooth, locally confined obstacle profile with pole singularities in the finite complex plane†, and the above conclusion regarding the role of nonlinear effects is generally valid.

The following discussion focuses on two particular obstacle shapes, (i) $f(x) = \operatorname{sech}^2 x$ and (ii) $f(x) = \operatorname{sech} x$, assuming that $A = O(1)$ so that nonlinear and dispersive effects are equally important. As indicated in Akylas & Yang (1995), these two examples typify obstacle profiles with pole singularities in the finite complex plane.

The asymptotic analysis parallels that of the model problem studied in Akylas & Yang (1995). However, the technical details turn out to be somewhat more involved here because the governing equation (2.2) is a partial differential equation and, generally, the nonlinear terms in (2.2) and in the boundary condition (2.3*a*) are higher than quadratic. In this section, we shall only outline the main steps in the analysis and give the final results; details of the derivation may be found in Appendices A and B.

† This excludes, for example, a Gaussian obstacle shape which has no singularities in the finite complex plane and requires separate treatment (Akylas & Yang 1995).

3.1. *Case (i)*: $f(x) = \mathrm{sech}^2 x$

For this obstacle profile,

$$\widehat{f}(k) = \tfrac{1}{2}k\,\mathrm{cosech}\frac{\pi k}{2}, \qquad \widehat{f^2}(k) = \tfrac{1}{12}k(k^2+4)\,\mathrm{cosech}\frac{\pi k}{2},$$

and $A = \epsilon/\mu^2$. The straightforward expansion (3.3) becomes disordered when $|k| = O(1/\mu) = O(\epsilon^{-1/2})$. To remove this non-uniformity, we replace (3.3) with the two-scale expansion

$$\widehat{\psi} = \mu\,\mathrm{cosech}\frac{\pi k}{2}\,\Phi(\kappa, z) + \cdots, \tag{3.4}$$

in terms of the scaled wavenumber variable $\kappa = \mu k$, with

$$\Phi \sim -\tfrac{1}{2}A\left\{\kappa q(z) - \kappa^3\left[\tfrac{1}{6}Aq'(0)q(z) + \sum_{r=1}^{\infty}\frac{1}{\kappa_r}\left(\tfrac{1}{6}A\delta_r - \gamma_r\right)\phi_r(z)\right] + \cdots\right\} \qquad (\kappa \to 0) \tag{3.5}$$

so that (3.4) is consistent with (3.3) when $1 \ll |k| \ll 1/\mu$.

As already remarked, $\widehat{\psi}(k,z)$, and hence $\Phi(\kappa,z)$, are expected to have simple-pole singularities on the real k-axis at the lee wavenumbers $k = \pm k_r$ ($\kappa = \pm\kappa_r^{1/2}$) ($r = 1, \ldots, N$). The dominant contribution to the lee-wave amplitude, however, comes from the residues of the poles that are closest to the origin. Accordingly, focusing attention on $\Phi(\kappa,z)$, we wish to compute the residues of the poles at $\kappa = \pm\kappa_N^{1/2}$ asymptotically as $\mu \to 0$ for $A = O(1)$.

Following the procedure described in Appendix A, upon substituting (3.4) in (3.1) and (3.2), it is found that, to leading order, Φ satisifies a Volterra integral–differential boundary-value problem. This problem is singular at $\kappa = \pm\kappa_N^{1/2}$, and posing the solution as a power series in κ consistent with (3.5), one may deduce the local behaviour of $\Phi(\kappa,z)$ near these singularities:

$$\Phi \sim -C_N\frac{\kappa}{\kappa^2 - \kappa_N}\phi_N(z) \qquad (\kappa \to \pm\kappa_N^{1/2}), \tag{3.6}$$

where $\phi_N(z)$ is the mode shape corresponding to the dominant lee-wave mode and C_N is a constant that can be determined numerically from an infinite sequence of boundary-value problems.

Combining (3.6) with (3.4), it is now seen that $\widehat{\psi}(k,z)$ has simple-pole singularities at $k = \pm k_N$ and the corresponding residues are known:

$$\widehat{\psi} \sim \mp C_N\,\frac{\exp\left(-\tfrac{1}{2}\pi k_N\right)}{k \mp k_N}\,\phi_N(z) \qquad (k \to \pm k_N).$$

Hence, returning to (2.4) and evaluating the contour integral, these poles contribute a lee wave with wavenumber k_N for $x > 0$:

$$\psi \sim 4\pi C_N \exp(-\tfrac{1}{2}\pi k_N)\phi_N(z)\sin k_N x \qquad (x \to \infty). \tag{3.7}$$

As expected, the lee-wave amplitude is exponentially small ($k_N = \kappa_N^{1/2}/\mu \gg 1$). In fact, comparing (3.7) with the linear result (2.10) (using $\widehat{f}(k_N) \sim k_N\exp(-\tfrac{1}{2}\pi k_N)$ in this case), it is seen that the effect of finite amplitude shows up solely in the value of the constant C_N; in order to assess the significance of finite-amplitude effects, one needs to know how C_N depends on the parameter A. This question is addressed in §5 by computing C_N numerically for specific examples.

3.2. *Case (ii): $f(x) = \operatorname{sech} x$*

In this case

$$\widehat{f}(k) = \tfrac{1}{2}\operatorname{sech}\frac{\pi k}{2}, \qquad \widehat{f^2}(k) = \tfrac{1}{2}k \operatorname{cosech}\frac{\pi k}{2},$$

and $A = \epsilon/\mu$. The straightforward expansion (3.3) becomes disordered when $k = O(1/\mu) = O(1/\epsilon)$, and, proceeding as in case (i), we propose the uniformly valid expansion

$$\widehat{\psi} = \mu \operatorname{sech}\frac{\pi k}{2}\,\Phi(\kappa, z) + \cdots \tag{3.8}$$

with

$$\Phi(\kappa, z) \sim -\tfrac{1}{2}Aq(z) + \tfrac{1}{2}A^2|\kappa| \left\{ q'(0)q(z) + \sum_{r=1}^{\infty} \frac{\delta_r}{\gamma_r}\phi_r(z) \right\} + \cdots \qquad (\kappa \to 0). \tag{3.9}$$

Again, we expect that $\Phi(\kappa, z)$ has simple-pole singularities at $\kappa = \pm\kappa_N^{1/2}$ and the goal is to compute the corresponding residues. Here, however, the (approximate) boundary-value problem that governs $\Phi(\kappa, z)$ to leading order for $|\kappa| = O(1)$ is not valid when κ is very close to these singularities ($\kappa_N^{1/2} - |\kappa| \leqslant O(\mu)$) and a local analysis is needed there. As explained in Akylas & Yang (1995) (see also Yang & Akylas 1995), this complication arises from the fact that $\Phi = O(1)$ as $\kappa \to 0$; this in turn is a consequence of the simple-pole singularities (closest to the real x-axis) of $\operatorname{sech} x$ which determine the dominant behaviour of $\widehat{f}(k)$ as $|k| \to \infty$.

Specifically, in the 'intermediate' region $\mu \ll \kappa_N^{1/2} - |\kappa| \ll 1$, where κ is close but not too close to $\pm\kappa_N^{1/2}$, the leading-order behaviour of $\Phi(\kappa, z)$ turns out to be (see Appendix B)

$$\Phi(\kappa, z) \sim \frac{C_N \phi_N(z)}{\left(\kappa_N^{1/2} - |\kappa|\right)^{\alpha}} \qquad (\mu \ll \kappa_N^{1/2} - |\kappa| \ll 1), \tag{3.10}$$

where C_N is a constant to be determined numerically, and

$$\alpha = 1 - \frac{A}{2\kappa_N^{1/2}}(\nu + \kappa_N \gamma_N \phi_N'(0)) \tag{3.11}$$

with

$$\nu = \int_0^1 \rho_0 \phi_N \left\{ \beta \mathcal{N}^2 q' \phi_N' - 2\frac{M_1}{F^2} q \phi_N + \beta M_1 (q\phi_N)' \right\} \mathrm{d}z$$

and

$$\gamma_N = \int_0^1 \rho_0 q \phi_N \mathrm{d}z.$$

Note that, according to (3.10), the singularities of $\Phi(\kappa, z)$ at $\kappa = \pm\kappa_N^{1/2}$ are not simple poles (save in the linear limit $A \to 0$), contrary to the fact that the induced lee wave has constant amplitude as $x \to \infty$; this is an indication that the asymptotic behaviour (3.10) breaks down in the immediate vicinity of $\kappa = \pm\kappa_N^{1/2}$, $\kappa_N^{1/2} - |\kappa| = O(\mu)$. Also, the argument used in Appendix B to obtain (3.10) is not strictly valid when $\alpha < 1$ – a different theoretical approach is needed but this matter is not pursued here. The physical significance of this condition is discussed in §5 for specific examples.

The correct structure of $\Phi(\kappa, z)$ at $\kappa = \pm\kappa_N^{1/2}$ can be obtained using a matched-asymptotics procedure in terms of the 'inner' variable $\eta = (\kappa_N^{1/2} - |\kappa|)/\mu$ (see Appendix

B). It transpires that $\Phi(\kappa, z)$ has simple-pole singularities at $\kappa = \pm\kappa_N^{1/2}$ and the corresponding residues are completely determined in terms of the constant C_N in (3.10). In view of (3.8), one then has

$$\hat{\psi} \sim \mp C_N \frac{2^\alpha}{\Gamma(\alpha)} \mu^{1-\alpha} \exp\left(\mp \mathrm{i}\tfrac{1}{2}\pi(\alpha-1)\right) \frac{\exp(-\frac{1}{2}\pi k_N)}{k \mp k_N} \phi_N(z) \qquad (k \to \pm k_N), \tag{3.12}$$

where $\Gamma(\alpha)$ denotes the gamma function.

Therefore, returning to (2.4) and accounting for the contribution of the poles of $\hat{\psi}$ at $k = \pm k_N$ according to (3.12), the induced lee wave downstream takes the form

$$\psi \sim 4\pi C_N \frac{2^\alpha}{\Gamma(\alpha)} \mu^{1-\alpha} \exp(-\tfrac{1}{2}\pi k_N)\, \phi_N(z) \sin(k_N x - \tfrac{1}{2}(\alpha-1)\pi) \qquad (x \to \infty). \tag{3.13}$$

Comparing the above expression with the corresponding linear result (2.10) (with $\hat{f}(k_N) \sim \exp(-\frac{1}{2}\pi k_N)$),

$$\psi \sim -2\pi A \gamma_N \kappa_N^{1/2} \exp(-\tfrac{1}{2}\pi k_N) \phi_N(z) \sin k_N x,$$

note that, apart from the value of C_N, here nonlinearity affects both the order of magnitude of the amplitude and the phase of the lee waves through the value of α which depends on the background flow and the nonlinearity parameter A according to (3.11). These additional nonlinear effects arise from the fact that the singularities (closest to the real axis) of the obstacle profile $f(x) = \operatorname{sech} x$ are simple poles so $\Phi(\kappa, z) = O(1)$ $(\kappa \to 0)$; this necessitates the local analysis near $\kappa = \kappa_N^{1/2}$ (see Appendix B), which determines the lee-wave amplitude and the phase shift that the lee wavetrain experiences owing to its interaction with the long wave above the topography. Similar effects are to be expected generally for lee waves induced by obstacle profiles with simple-pole singularities such as, for example, the algebraic obstacle or 'Witch of Agnesi' (Akylas & Yang 1995; Yang & Akylas 1995).

4. Numerical solution

The asymptotic results (3.7) and (3.13) suggest that, for $A = O(1)$, nonlinearity may seriously affect the lee-wave pattern induced by a long smooth obstacle. On the other hand, the asymptotic theory is formally valid in the weakly nonlinear–weakly dispersive limit $(\epsilon, \mu \ll 1)$ where lee waves have exponentially small amplitude and form only a small portion of the overall disturbance. From a more practical point of view, therefore, it would be of interest to know whether nonlinear effects play an equally important part in the parameter range where the lee-wave amplitude is more substantial. To address this issue, we shall resort to a numerical procedure for calculating steady, finite-amplitude lee waves which also provides independent confirmation of the asymptotic theory.

One of the complications in the numerical treatment of the nonlinear lee-wave problem (2.2)–(2.3) arises from the lower boundary condition (2.3a) that holds on the bottom bump. Although this condition can be implemented through a topography-following coordinate transformation (see, for example, Lilly & Klemp 1979), we find it more convenient to work in $0 \leqslant z \leqslant 1$ by applying the 'fictitious' condition

$$\psi(x, 0) = -\epsilon \bar{f}(x) \tag{4.1}$$

at $z = 0$, rather than the actual nonlinear condition (2.3a) at $z = \epsilon f(x)$ (the same device was used by Laprise & Peltier 1989). Of course, $\bar{f}(x)$ has to be such that (2.3a)

is also satisfied and this is achieved via an iterative procedure: denoting by $\bar{f}^{(n)}(x)$ the estimate of $\bar{f}(x)$ and by $\psi^{(n)}(x,z)$ the corresponding estimate of $\psi(x,z)$ after n iterations, the next iteration for determining $\psi^{(n+1)}$ proceeds using

$$\bar{f}^{(n+1)}(x) = \bar{f}^{(n)}(x) + f(x) + \frac{1}{\epsilon}\psi^{(n)}\left(x, \epsilon f(x)\right),$$

where $f(x)$ is the obstacle profile. (The iteration scheme is initiated by choosing $\bar{f}^{(1)}(x) = f(x)$ in which case (4.1) reduces to the linear boundary condition.)

Attention now is focused on the solution of equation (2.2) in $0 \leqslant z \leqslant 1$, $-\infty < x < \infty$, subject to the upstream condition (2.1) and the boundary conditions (2.3*b*) and (4.1), bearing in mind that $\bar{f}(x)$ and $\psi(x,z)$ are intermediate estimates in the above iteration scheme. Mathematically, the governing equation (2.2) is an elliptic partial differential equation while the boundary conditions are of the parabolic type in the sense that a downstream condition cannot be specified *a priori*. For this reason, numerical-instability problems are to be expected if one attempts to determine the lee waves induced downstream by solving (2.2) through a marching scheme, starting from far upstream.

To circumvent this difficulty, in analogy with the linearized response (2.5), we expand $\psi(x,z)$ in terms of the linear hydrostatic response $q(z)$ (see (2.6*b*)) and the linear internal-wave modes defined in (2.7*b*):

$$\psi(x,z) = -\epsilon\bar{f}(x)q(z) + \sum_{r=1}^{\infty} a_r(x)\phi_r(z). \tag{4.2}$$

Hence, the boundary conditions (2.3*b*) and (4.1) are automatically met, and substituting (4.2) into (2.2), making use of the orthogonality relation (2.8), yields an (infinite) system of coupled nonlinear ordinary differential equations for the modal amplitudes $a_r(x)$ $(r = 1, 2, \ldots)$.

In this system, the equations for the amplitudes of the modes that contribute lee waves downstream $(r = 1, \ldots, N)$ are solved numerically by a standard fourth-order Runge–Kutta marching scheme starting far upstream; the equations for the remaining evanescent modes are treated as boundary-value problems by imposing the downstream conditions

$$a_r \to 0 \qquad (x \to \infty;\ r = N+1, N+2, \ldots).$$

Of course, since the system of equations at hand is nonlinear and coupled, it is necessary to use an (inner-level) iteration procedure in order to solve for the modal amplitudes. However, in the examples discussed below the nonlinear coupling among the modes happens to be relatively weak, so typically 5–10 Jacobi iterations are sufficient to obtain converged solutions for a_r. After these amplitudes are found, the 'fictitious' obstacle profile $\bar{f}(x)$ is updated as already described, and the procedure is repeated until the exact lower boundary condition (2.3*a*) is met.

Finally, in implementing this numerical procedure, it is necessary to use a finite computational domain in the streamwise direction and to truncate the modal expansion (4.2) to a finite number of modes. The size of the computational domain is chosen such that the height of the obstacle at the boundaries is small compared to the lee-wave amplitude. Also, in the examples discussed below, typically, using eight modes is sufficient to compute lee waves with $O(10^{-10})$ amplitude as long as the grid size (used in computing the modes $\phi_r(z)$ and the modal amplitudes $a_r(x)$) is also made fine enough to be consistent with this accuracy.

5. Results and discussion

In the following discussion, the density stratification is taken to be such that $\beta = 0$ (Boussinesq approximation, $\rho_0 = 1$) and $\mathcal{N}^2(z)$ varies linearly with height:

$$\mathcal{N}^2(z) = 1 + bz \qquad (0 \leqslant z \leqslant 1),$$

where $b \geqslant 0$ is a constant, so $M_j \equiv 0$ ($j \geqslant 2$) in (3.1) and the problem simplifies considerably. Moreover, it is assumed that $c_2 < F < c_1$ so lee waves of the first mode only are excited ($N = 1$).

Under these conditions, we shall present asymptotic and numerical results for lee waves induced by obstacles having either a $\operatorname{sech}^2 x$ or $\operatorname{sech} x$ profile, and for two particular examples of density stratification, namely $b = 0$ and $b = 1$.

5.1. *Uniform stratification*

When $b = 0$, we are dealing with a uniformly stratified Boussinesq fluid (constant $\mathcal{N}$, $\beta = 0$) in which case Long's model applies and the governing equation (2.2) is linear (of course the bottom boundary condition (2.3*a*) still is nonlinear). Furthermore, the eigenvalue problems (2.7*b*), (2.9*b*) and the boundary-value problem (2.6*b*) for the hydrostatic response can be solved in closed form:

$$\kappa_r = \frac{1}{F^2} - r^2\pi^2, \qquad \phi_r(z) = \sqrt{2}\sin r\pi z \quad (r = 1, 2, \ldots);$$

$$c_s = \frac{1}{s\pi}, \qquad f_s(z) = \sqrt{2}\sin s\pi z \quad (s = 1, 2, \ldots);$$

and

$$q(z) = \frac{\sin(1-z)/F}{\sin(1/F)}.$$

Also, from (3.11),

$$\alpha = 1 + \frac{\pi^2}{\kappa_1^{1/2}} A \tag{5.1}$$

and $\alpha > 1$ always. According to the criterion given in Grimshaw & Yi (1991), non-resonant subcritical flow (relative to c_1) occurs when $c_2 + \epsilon\Delta_2 < F < c_1 - \epsilon\Delta_1$ with $\Delta_1 = \frac{1}{2}\sqrt{6}\pi^{-5/2} = 0.07$, $\Delta_2 = \frac{1}{8}\sqrt{3}\pi^{-5/2} = 0.01$. We choose $1/F^2 = \pi^2 + 2$ ($F = 0.2903 = c_1 - 0.028$); this value of the flow speed is within the non-resonant range for the values of ϵ considered here ($\epsilon < 0.1$), and the assumption of steady flow is expected to hold.

As noted in §3, the asymptotic expressions (3.7) and (3.13) for the lee wavetrain downstream depend on the constant C_1 ($N = 1$) which needs to be determined numerically. Specifically, the value of C_1 can be estimated from the coefficients $b_m(z)$ of the power series (A 3) (for $f = \operatorname{sech}^2 x$) and (B 11) (for $f = \operatorname{sech} x$) by computing the projection of $b_m(z)$ on $\phi_1(z)$ as $m \to \infty$. Note that for $f = \operatorname{sech}^2 x$, $b_m(z)$ ($m \geqslant 2$) satisfy the sequence of boundary-value problems (A 4)–(A 5*b*), while for $f = \operatorname{sech} x$, an analogous sequence is obtained upon substitution of (B 11) into (B 1)–(B 2*b*). In solving for $b_m(z)$ numerically, it is important to note that, as m increases, high-order derivatives of $b_1, b_2, \ldots$ enter the inhomogeneous boundary condition (A 5*a*); these need to be evaluated accurately to avoid introducing significant error in estimating C_1. (Details of the numerical procedure for determining C_1 can be found in Yang 1995.)

Table 1 lists values of the constant C_1 for certain values of A. It is worth noting that, as the nonlinearity parameter A is increased, C_1 increases more rapidly than linear theory would imply; for example, doubling the obstacle height can result in an

$f(x) = \text{sech}^2 x$		$f(x) = \text{sech}\, x$	
A	C_1	A	C_1
$A \to 0$	$2^{-1/2}\pi A$	$A \to 0$	$\frac{1}{4}\pi A$
0.2	1.3	0.1	0.20
0.4	6.3	0.2	0.99
0.8	52		

TABLE 1. Computed values of the constant C_1 appearing in the asymptotic results (3.7) and (3.13) for certain values of the parameter A in the case of uniformly stratified flow of a Boussinesq fluid ($\mathcal{N} = 1$, $\beta = 0$) over the two obstacle shapes $f(x) = \text{sech}^2 x$ and $f(x) = \text{sech}\, x$. The Froude number $F = 0.2903$.

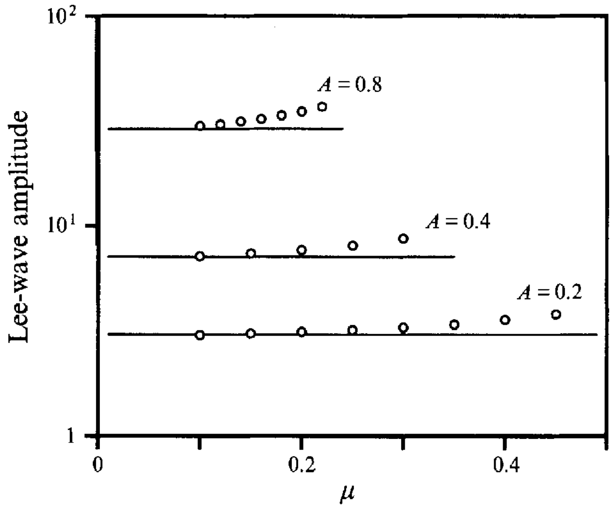

FIGURE 1. The lee-wave amplitude (normalized with the result of linear theory (2.10)) as a function of the dispersion parameter μ for uniformly stratified flow of a Boussinesq fluid ($\mathcal{N} = 1$, $\beta = 0$) over the obstacle $f(x) = \text{sech}^2 x$, and for certain values of the parameter A: ——, asymptotic results (3.7); ○, numerical results. The Froude number $F = 0.2903$.

order-of-magnitude increase of the lee-wave amplitude. This suggests that the range of validity of the linear theory is limited.

We now turn to a discussion of results from a fully numerical computation of lee waves. In the case of uniformly stratified flow, as noted earlier, the governing equation is linear. Hence, the equations for the modal amplitudes $a_r(x)$ in (4.2) are uncoupled and can be readily integrated without iteration. Figures 1 and 2(*a*), respectively, show plots of the computed lee-wave amplitude (defined as the amplitude of $a_1(x)$ as $x \to \infty$) against the dispersion parameter μ (keeping A fixed) for the two obstacle shapes $f(x) = \text{sech}^2 x$ and $f(x) = \text{sech}\, x$. Figure 2(*b*) shows the lee-wave phase shift, caused by nonlinearity, as a function of μ for $f(x) = \text{sech}\, x$. In these plots, the predictions of the asymptotic theory (viz. (3.7) and (3.13) respectively) are put together for comparison. Also, to bring out the effect of finite amplitude, several values of the parameter A are considered and the results are normalized by the corresponding linear result (2.10).

According to figures 1 and 2, there is reasonably good agreement between the asymptotic and numerical results for a wide range of values of μ, well beyond the

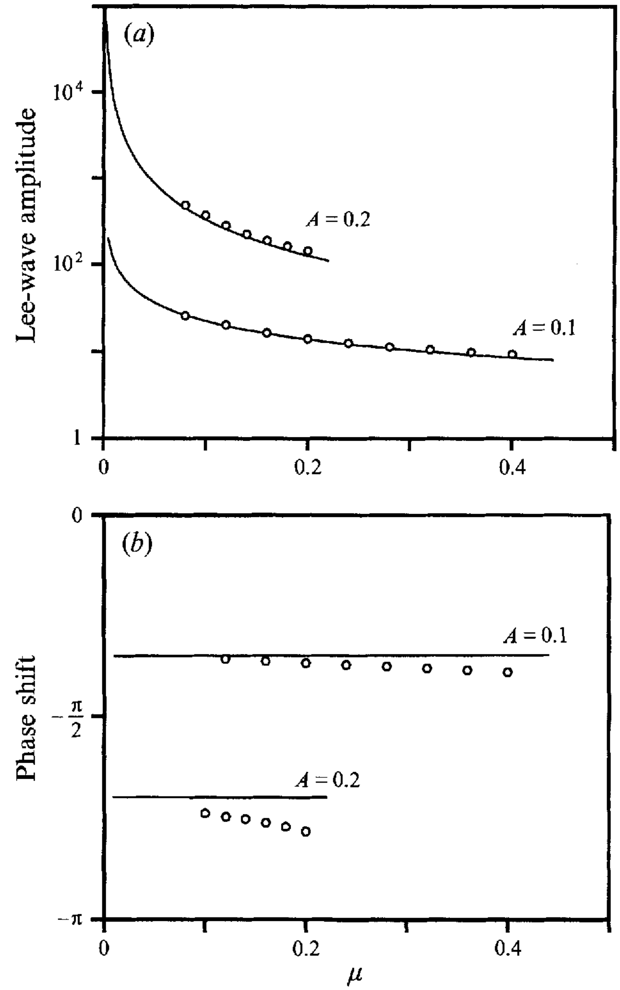

FIGURE 2. Asymptotic (——) and numerical (o) results as a function of the dispersion parameter μ for lee waves induced by uniformly stratified flow of a Boussinesq fluid ($\mathcal{N} = 1$, $\beta = 0$) over the obstacle $f(x) = \operatorname{sech} x$, and for certain values of the parameter A: (a) the lee-wave amplitude (normalized with the result of linear theory (2.10)); (b) the lee-wave phase shift. The Froude number $F = 0.2903$.

weakly nonlinear, nearly hydrostatic régime ($\epsilon, \mu \ll 1$) where the theory is formally valid; in fact, the flows corresponding to the largest values of μ for which results are plotted in figures 1 and 2 are quite nonlinear as regions of closed streamlines are about to appear (when this happens, Long's equation (2.2) is invalidated because not all streamlines originate far upstream). Moreover, as suggested by the asymptotic theory, the effect of finite amplitude results in significant amplification of the lee waves, even for values of A as small as 0.1.

Comparing figure 2(a) with figure 1, it is clear that, in the limit $\mu \to 0$, finite-amplitude effects are relatively stronger for $f = \operatorname{sech} x$; the ratio of the nonlinear to the linear lee-wave amplitude tends to ∞ as $\mu \to 0$ in figure 2(a) because the value of α in (3.13) is always greater than 1 as noted in (5.1). The actual lee-wave amplitude, however, is negligibly small for both $f = \operatorname{sech}^2 x$ and $f = \operatorname{sech} x$ in this limit, and the difference between these two obstacle profiles has no serious physical consequences.

On the other hand, when μ is not very small, the lee-wave amplitude can be

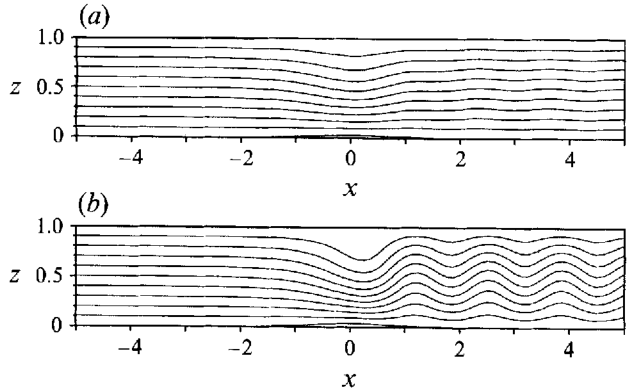

FIGURE 3. Streamlines of uniformly stratified flow of a Boussinesq fluid ($\mathcal{N} = 1$, $\beta = 0$) over the obstacle $f(x) = \operatorname{sech}^2 x$ for $\mu = 0.3$, $A = 0.4$ ($\epsilon = 0.036$): (*a*) results of linear theory; (*b*) numerically computed results on the basis of the full nonlinear equations. The Froude number $F = 0.2903$.

substantial and, physically, the role of nonlinearity in enhancing the lee waves induced by both obstacle shapes is of far greater significance. This point is clearly demonstrated in figure 3 which shows streamlines† of the flow field induced by the obstacle $f(x) = \operatorname{sech}^2 x$ for $\mu = 0.3$, $A = 0.4$ ($\epsilon = 0.036$), as predicted by the linearized equations of motion (figure 3*a*) and the full nonlinear equations (figure 3*b*). Even though the value of ϵ is quite small, linear theory grossly underestimates the lee waves downstream. On the basis of the asymptotic theory, this dramatic difference can be attributed to the fact that, for $A = 0.4$, the constant C_1 in (3.7) is about 7 times larger than the corresponding linear ($A \rightarrow 0$) value (see table 1).

We also remark that, unlike their completely different behaviour in the limit $\mu \rightarrow 0$, the lee waves induced by the two obstacle profiles considered here behave similarly for moderately small values of μ, when the lee-wave amplitude is appreciable. To illustrate this, for $\mu = 0.3\pi/2 = 0.47$, $\epsilon = 0.036$ ($A = 0.076$), the profile $f(x) = \operatorname{sech} x$ closely resembles $f(x) = \operatorname{sech}^2 x$ for $\mu = 0.3$, $\epsilon = 0.036$ ($A = 0.4$) – both have the same peak amplitude and enclose the same area. Under these conditions, the two obstacle profiles give rise to very similar lee-wave patterns; in both cases the nonlinear response is several times larger than the linear one, as illustrated in figure 3 for $f(x) = \operatorname{sech}^2 x$.

5.2. *Linearly varying* $\mathcal{N}^2$

In the case that $\mathcal{N}^2(z)$ varies linearly with depth ($b = 1$), it is necessary to solve the problems (2.6*b*), (2.7*b*) and (2.9*b*) numerically; moreover, depending on the flow speed, the value of α in (3.13) can be less than 1.

From (2.9*b*), we find $c_1 = 0.391$, $c_2 = 0.194$, and as a first choice of flow speed we take $F = 0.25$. According to Grimshaw & Smyth (1986), this value of F is within the non-resonant range $c_2 + \epsilon^{1/2}\Delta_2 < F < c_1 - \epsilon^{1/2}\Delta_1$ ($\Delta_1 = 0.22$, $\Delta_2 = 0.03$) for $\epsilon < 0.2$. The eigenvalue problem (2.7*b*) then yields $\kappa_1 = 14.4$ and one has $\alpha = 1 + 0.12A$ from (3.11). Computed values of the constant C_1 under these flow conditions are listed in table 2, for the two obstacle shapes under consideration and certain values of the nonlinearity parameter A. As in the case of uniform stratification, C_1 increases with A

† The streamline pattern shown in figure 3(*a*) satisfies the linearized version of the boundary condition (2.3*a*) that is applied on $z = 0$, so the first streamline above $z = 0$ does not coincide with the obstacle profile exactly. On the other hand, the nonlinear response shown in figure 3(*b*) satisfies the nonlinear boundary condition (2.3*a*) on $z = \epsilon f(x)$, and the first streamline above $z = 0$ coincides with the obstacle profile.

$f(x) = \mathrm{sech}^2 x$		$f(x) = \mathrm{sech}\, x$	
A	C_1	A	C_1
$A \to 0$	$1.8A$	$A \to 0$	$0.24A$
0.2	0.49	0.1	0.034
0.4	2.0	0.2	0.15

TABLE 2. Computed values of the constant C_1 appearing in the asymptotic results (3.7) and (3.13) for certain values of the parameter A in the case of stratified flow of a Boussinesq fluid ($\beta = 0$) with $\mathcal{N}^2 = 1 + z$ over the two obstacle shapes $f(x) = \mathrm{sech}^2 x$ and $f(x) = \mathrm{sech}\, x$. The Froude number $F = 0.25$.

more rapidly than linear theory would predict so nonlinearity is expected to amplify the lee waves.

When the stratification is not uniform, there is an additional technical complication in computing finite-amplitude lee waves because the governing equation (2.2) is nonlinear and it becomes necessary to use iteration in solving for the modal amplitudes $a_r(x)$ in (4.2). Nevertheless, as shown in figures 4(a) and 4(b) for $F = 0.25$, the results are qualitatively similar to the case of constant $\mathcal{N}$ discussed above (see figures 1, 2a). Again nonlinearity increases the lee-wave amplitude significantly, although not as dramatically as in the previous case, consistent with the asymptotic theory. Also, for $f(x) = \mathrm{sech}^2 x$, as A is increased, the validity of the asymptotic results seems to be limited to a relatively narrow region close to $\mu = 0$.

Finally, we consider the flow speed $F = 0.3162$ ($F^2 = 0.1$) which also lies within the subcritical non-resonant range for $\epsilon < 0.1$, say. For this value of F, (2.7b) yields $\kappa_1 = 5.24$ and $\alpha = 1 - 0.85A$ according to (3.11). Hence $\alpha < 1$ and, even though the asymptotic analysis in §3.2 that leads to (3.13) for $f(x) = \mathrm{sech} x$ is not strictly valid under these flow conditions (see Appendix B), we suspect that here nonlinearity could diminish the lee-wave amplitude. As shown in figure 5, this hypothesis seems to be supported by numerical computations only when μ and ϵ are quite small and the lee-wave amplitude is negligible. For larger values of μ and ϵ, the computed lee-wave amplitudes still are somewhat greater than those predicted by linear theory, but the effect of nonlinearity is not as pronounced as in the cases discussed earlier where $\alpha > 1$.

6. Concluding remarks

We have studied the effect of nonlinearity on steady lee-wave patterns induced by subcritical stratified flow over smooth topography. For a given topography shape, the relative significance of nonlinearity in comparison with dispersion is measured by the parameter A defined in (1.1), and the classical linear lee-wave theory is valid only when $A \to 0$. For $A = O(1)$, when weak nonlinear and dispersive effects are equally important, the amplitude of the induced lee wave, even though it is exponentially small, generally is determined by a fully nonlinear mechanism. In this régime, we have developed an asymptotic theory using asymptotics beyond all orders, which reveals that nonlinearity can enhance the lee-wave amplitude dramatically. We have also carried out numerical computations to confirm the predictions of the asymptotic theory. Our computations, in addition, indicate that the asymptotic results remain reasonably accurate for a wide range of flow conditions beyond the formal region

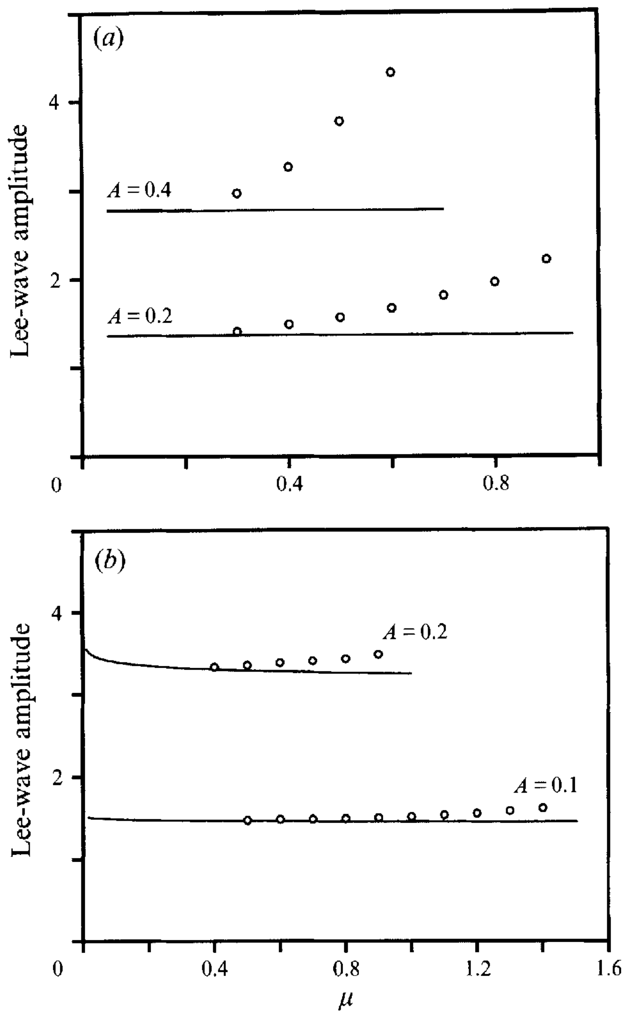

FIGURE 4. The lee-wave amplitude (normalized with the result of linear theory (2.10)) as a function of the dispersion parameter μ for stratified flow of a Boussinesq fluid ($\beta = 0$) with $\mathcal{N}^2 = 1 + z$ and for certain values of the parameter A: ——, asymptotic results (3.7); ○ numerical results. The Froude number $F = 0.25$. (*a*) Flow over the obstacle $f(x) = \operatorname{sech}^2 x$; and (*b*) over the obstacle $f(x) = \operatorname{sech} x$.

of validity of the theory. Under such conditions, the (formally exponentially small) lee-wave amplitude is substantial and can be significantly larger, sometimes by an order of magnitude, than the estimate obtained on the basis of linear lee-wave theory.

In the present study, we have considered steady flow, excluding the possibility of upstream influence. We expect these conditions to hold as long as the flow speed is not close to critical and no wave breaking (density inversions) occurs during the transient development of the flow (Grimshaw & Smyth 1986; Grimshaw & Yi 1991; Lamb 1994). Furthermore, the inviscid model used here is expected to be relevant as long as no flow separation occurs. In spite of these limitations of the theory, the effect of nonlinearity on the induced lee waves is very pronounced even when the topography has small amplitude, and should be noticeable in experiments.

Finally, it would be worth exploring finite-amplitude effects on three-dimensional lee-wave patterns induced by stratified flow over topography that depends on both

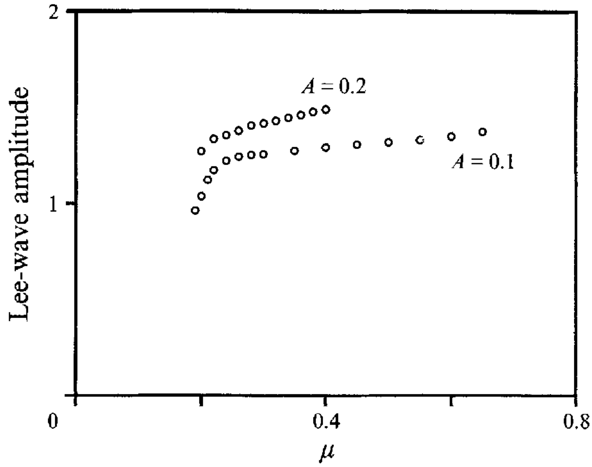

FIGURE 5. Numerically computed values of the lee-wave amplitude (normalized with the result of linear theory (2.10)) as a function of the dispersion parameter μ for stratified flow of a Boussinesq fluid ($\beta = 0$) with $\mathscr{N}^2 = 1 + z$ over the obstacle $f(x) = \operatorname{sech} x$, and for certain values of the parameter A. The Froude number $F = 0.3162$.

the streamwise and the spanwise directions. In this case, a continuous distribution of wavenumbers is excited, and nonlinearity may influence certain portions of the wave pattern more seriously than others.

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Appendix A. Asymptotic analysis for $f(x) = \operatorname{sech}^2 x$

Here we give details of the asymptotic analysis that leads to expression (3.7) for the lee wave induced by the topography $f(x) = \operatorname{sech}^2 x$.

Substituting (3.4) into (3.1), it is found that, to leading order, Φ satisfies the integral–differential equation

$$\begin{aligned}
&\Phi_{zz} - \beta\mathscr{N}^2\Phi_z + \left(\frac{\mathscr{N}^2}{F^2} - \kappa^2\right)\Phi \\
&= \beta\mathscr{N}^2\left\{\int_0^\kappa \mathrm{d}\lambda\ [\Phi_z(\kappa-\lambda,z)\Phi_z(\lambda,z) - \lambda(\kappa-\lambda)\Phi(\kappa-\lambda,z)\Phi(\lambda,z)]\right\} \\
&\quad - \sum_{j=1}^{\infty} 2^j M_j \left\{\frac{Y_{j+1}}{F^2} - \beta\int_0^\kappa \mathrm{d}\lambda\ Y_j(\kappa-\lambda,z)\int_0^\lambda \mathrm{d}\lambda_1 \Phi_z(\lambda-\lambda_1,z)\Phi_z(\lambda_1,z)\right. \\
&\quad \left. - \frac{\beta}{j+1}(Y_{j+1})_z + \beta\int_0^\kappa \mathrm{d}\lambda\ Y_j(\kappa-\lambda,z)\int_0^\lambda \mathrm{d}\lambda_1\ (\lambda-\lambda_1)\lambda_1\Phi(\lambda-\lambda_1,z)\Phi(\lambda,z)\right\},
\end{aligned} \tag{A 1}$$

with the notation

$$Y_j(\kappa, z) = \int_0^{\kappa} \mathrm{d}\lambda_1\, \Phi(\kappa - \lambda_1, z) \int_0^{\lambda_1} \mathrm{d}\lambda_2\, \Phi(\lambda_1 - \lambda_2, z)$$
$$\cdots \int_0^{\lambda_{j-2}} \mathrm{d}\lambda_{j-1}\, \Phi(\lambda_{j-2} - \lambda_{j-1}, z)\Phi(\lambda_{j-1}, z).$$

Moreover, the boundary conditions (3.2) yield

$$\Phi(\kappa, 0) + \sum_{j=1}^{\infty} \frac{A^j}{j!(2j-1)!} \int_0^{\kappa} \mathrm{d}\lambda\, \frac{\partial^j \Phi}{\partial z^j}(\kappa - \lambda, 0)\lambda^{2j-1} = -\tfrac{1}{2}A\kappa, \qquad \text{(A 2a)}$$

$$\Phi(\kappa, 1) = 0. \qquad \text{(A 2b)}$$

The solution of this problem is posed as a power series,

$$\Phi(\kappa, z) = \sum_{m=1}^{\infty} b_m(z)\kappa^{2m-1}, \qquad \text{(A 3)}$$

with $b_1 = -\tfrac{1}{2}Aq(z)$ according to (3.5). The coefficients $b_m (m \geqslant 2)$ then can be determined successively from the infinite sequence of boundary-value problems

$$\begin{aligned} b_m'' - \beta\mathcal{N}^2 b_m' + \frac{\mathcal{N}^2}{F^2} b_m &= b_{m-1} + \beta\mathcal{N}^2(L_m - S_m) \\ &\quad - \sum_{j=1}^{m-1} 2^j M_j \left\{ \frac{P_m^{(j+1)}}{F^2} - \frac{\beta}{j+1}\left(P_m^{(j+1)}\right)' \right\} \\ &\quad - \beta\sum_{j=1}^{m-3} 2^j M_j Q_m^{(j)} + \beta\sum_{j=1}^{m-2} 2^j M_j R_m^{(j)} \qquad (0 \leqslant z \leqslant 1), \end{aligned} \qquad \text{(A 4)}$$

$$b_m(0) = -\sum_{j=1}^{m-1} A^j \frac{(2m-2j-1)!}{j!(2m-1)!} \frac{\mathrm{d}^j b_{m-j}}{\mathrm{d}z^j}(0), \qquad \text{(A 5a)}$$

$$b_m(1) = 0. \qquad \text{(A 5b)}$$

The expressions for L_m, S_m, $P_m^{(j)}$, $Q_m^{(j)}$ and $R_m^{(j)}$ above contain complicated convolution sums involving $b_1, \ldots, b_{m-1}$; they arise from the multiple convolution integrals in (A 1) and are given by

$$L_m = \sum_{s=1}^{m-1} \frac{(2m-2s-1)!(2s-1)!}{(2m-1)!} b'_{m-s} b'_s;$$

$$S_m = \sum_{s=1}^{m-2} \frac{(2m-2s-2)!(2s)!}{(2m-1)!} b_{m-s-1} b_s \quad (m \geqslant 3), \quad S_2 = 0;$$

$$S_m = \sum_{s=1}^{m-2} \frac{(2m-2s-2)!(2s)!}{(2m-1)!} b_{m-s-1} b_s \quad (m \geqslant 3), \quad S_2 = 0;$$

$$P_m^{(1)} = b_m, \quad P_m^{(j+1)} = \sum_{s=j}^{m-1} \frac{(2m-2s-1)!(2s-1)!}{(2m-1)!} b_{m-s} P_s^{(j)};$$

$$Q_m^{(j)} = \sum_{s=j}^{m-3} \frac{(2m-2s-1)!(2s-1)!}{(2m-1)!} S_{m-s} P_s^{(j)} \ (m \geqslant 4), \quad Q_2^{(j)} = Q_3^{(j)} = 0;$$

$$R_m^{(j)} = \sum_{s=j}^{m-2} \frac{(2m-2s-1)!(2s-1)!}{(2m-1)!} L_{m-s} P_s^{(j)} \ (m \geqslant 3), \quad R_2^{(j)} = 0.$$

In particular, for $m = 2$, (A 4)–(A 5b) give

$$b''_2 - \beta \mathcal{N}^2 b'_2 + \frac{\mathcal{N}^2}{F^2} b_2 = b_1 + \tfrac{1}{6}\beta \mathcal{N}^2 b_1'^2 + \tfrac{1}{3} M_1 \left(\beta b_1 b'_1 - \frac{b_1^2}{F^2} \right) \qquad (0 \leqslant z \leqslant 1),$$

$$b_2(0) = -\tfrac{1}{6} A b'_1(0), \qquad b_2(1) = 0;$$

the solution is

$$b_2 = \tfrac{1}{12} A^2 q'(0) q(z) + \tfrac{1}{2} A \sum_{r=1}^{\infty} \frac{1}{\kappa_r} \left(\tfrac{1}{6} A \delta_r - \gamma_r \right) \phi_r(z),$$

consistent with (3.5).

Although, in principle, one may proceed to determine $b_m(z)$ for $m > 2$, it is the asymptotic behaviour of $b_m(z)$ as $m \to \infty$ that controls the convergence of the power series (A 3) and, hence, the singularities of $\Phi(\kappa, z)$ in the κ-plane. In the limit $m \to \infty$, the boundary-value problem (A 4)–(A 5b) in fact simplifies significantly because the contribution of the convolution sums in (A 4) is subdominant,

$$b''_m - \beta \mathcal{N}^2 b'_m + \frac{\mathcal{N}^2}{F^2} b_m \sim b_{m-1} \qquad (m \to \infty),$$

and the boundary condition (A 5a) becomes effectively homogeneous:

$$b_m(0) \to 0 \qquad (m \to \infty).$$

Hence, $b_m(z)$ may be expanded in terms of the lee-wave modes defined by the eigenvalue problem (2.7b):

$$b_m(z) \sim \sum_{r=1}^{\infty} g_{mr} \phi_r(z) \qquad (m \to \infty).$$

The coefficients g_{mr} obey the simple recurrence relation

$$\kappa_r g_{mr} \sim g_{m-1r},$$

κ_r being the corresponding eigenvalues, so that

$$g_{mr} \sim C_r \kappa_r^{-m}$$

and

$$b_m \sim \sum_{r=1}^{\infty} C_r \kappa_r^{-m} \phi_r(z) \qquad (m \to \infty), \tag{A 6}$$

where C_r are certain constants that depend on $b_1(z)$ and cannot be determined by asymptotic analysis alone.

Combining (A 6) with (A 3), it is now clear that $\Phi(\kappa, z)$ has simple-pole singularities at $\kappa = \pm\kappa_N^{1/2}$:

$$\Phi \sim -C_N \frac{\kappa}{\kappa^2 - \kappa_N} \phi_N(z) \qquad (\kappa \to \pm\kappa_N^{1/2}).$$

Appendix B. Asymptotic analysis for $f(x) = \operatorname{sech} x$

Here we give details of the asymptotic analysis that leads to expression (3.13) for the lee wave induced by the topography $f(x) = \operatorname{sech} x$.

Assuming for now that κ is not too close to $\pm\kappa_N^{1/2}$, upon substituting (3.8) into (3.1)–(3.2), $\Phi(\kappa, z)$ satisfies to leading order

$$\begin{aligned}
&\Phi_{zz} - \beta\mathcal{N}^2\Phi_z + \left(\frac{\mathcal{N}^2}{F^2} - \kappa^2\right)\Phi \\
&\quad = \beta\mathcal{N}^2 \operatorname{sgn}\kappa \left\{\int_0^\kappa \mathrm{d}\lambda\, [\Phi_z(\kappa-\lambda, z)\Phi_z(\lambda, z) - \lambda(\kappa-\lambda)\Phi(\kappa-\lambda, z)\Phi(\lambda, z)]\right\} \\
&\quad\quad - \sum_{j=1}^{\infty} 2^j M_j \left\{\frac{\widetilde{Y}_{j+1}}{F^2} - \beta\int_0^\kappa \mathrm{d}\lambda\, \widetilde{Y}_j(\kappa-\lambda, z)\int_0^\lambda \mathrm{d}\lambda_1\, \Phi_z(\lambda-\lambda_1, z)\Phi_z(\lambda_1, z)\right. \\
&\quad\quad \left. - \frac{\beta}{j+1}(\widetilde{Y}_{j+1})_z + \beta\int_0^\kappa \mathrm{d}\lambda\, \widetilde{Y}_j(\kappa-\lambda, z)\int_0^\lambda \mathrm{d}\lambda_1 (\lambda-\lambda_1)\lambda_1 \Phi(\lambda-\lambda_1, z)\Phi(\lambda_1, z)\right\},
\end{aligned} \tag{B 1}$$

subject to the boundary conditions

$$\Phi(\kappa, 0) + \sum_{j=1}^{\infty} \frac{A^j (\operatorname{sgn}\kappa)^j}{j!(j-1)!} \int_0^\kappa \mathrm{d}\lambda\, \frac{\partial^j \Phi}{\partial z^j}(\kappa - \lambda, 0)\lambda^{j-1} = -\tfrac{1}{2}A, \tag{B 2a}$$

$$\Phi(\kappa, 1) = 0, \tag{B 2b}$$

where $\widetilde{Y}_j(\kappa, z) = (\operatorname{sgn}\kappa)^{j-1} Y_j(\kappa, z)$ in terms of the $Y_j(\kappa, z)$ appearing in (A 1).

In preparation for the local analysis that becomes necessary close to $\pm\kappa_N^{1/2}$, next we examine the asymptotic behaviour of $\Phi(\kappa, z)$ in the 'intermediate' region where κ is close to $\pm\kappa_N^{1/2}$ but (B 1)–(B 2b) still hold, $\mu \ll \kappa_N^{1/2} - |\kappa| \ll 1$. Specifically, suppose

$$\Phi(\kappa, z) \sim \frac{C_N \phi_N(z)}{\left(\kappa_N^{1/2} - |\kappa|\right)^{\alpha}} + \frac{C_0 q(z)}{\left(\kappa_N^{1/2} - |\kappa|\right)^{\alpha-1}} + \cdots \qquad (\mu \ll \kappa_N^{1/2} - |\kappa| \ll 1), \tag{B 3}$$

where C_N, C_0 and α are to be determined. Then, by dominant balance, it follows from (B 1) and (B 2*a*) respectively that

$$2C_N\kappa_N^{1/2} - C_0\kappa_N\gamma_N = \frac{A\nu}{1-\alpha}C_N, \tag{B 4a}$$

$$C_0 = \frac{A\phi_N'(0)}{1-\alpha}C_N, \tag{B 4b}$$

where

$$\nu = \int_0^1 \rho_0\phi_N\left\{\beta\mathcal{N}^2 q'\phi_N' - 2\frac{M_1}{F^2}q\phi_N + \beta M_1(q\phi_N)'\right\}\mathrm{d}z.$$

Hence,

$$\alpha = 1 - \frac{A}{2\kappa_N^{1/2}}\left(\nu + \kappa_N\gamma_N\phi_N'(0)\right), \tag{B 5}$$

and, on the basis of (B 1),(B 2*b*), the singularities of $\Phi(\kappa, z)$ at $\kappa = \pm\kappa_N^{1/2}$ are not simple poles (save in the linear limit $A \to 0$), contrary to the fact that the induced lee wave has constant amplitude as $x \to \infty$; this is an indication that the leading-order problem (B 1)–(B 2*b*) for Φ breaks down in the immediate vicinity of $\kappa = \pm\kappa_N^{1/2}$, $\kappa_N^{1/2} - |\kappa| = O(\mu)$, and the asymptotic behaviour (B 3) is not valid there. Also note that the dominant-balance argument used above to obtain (B 4*b*) is not valid when $\alpha < 1$ – a different theoretical approach is needed but this matter is not pursued here.

To obtain the structure of $\widehat{\varphi}(k, z)$ close to $k = \pm k_N$ (assuming $\alpha > 1$), we return to the (exact) boundary-value problem (3.1)–(3.2) and approximate the terms involving convolution integrals more accurately. Specifically, near $\kappa = \kappa_N^{1/2}$, (B 3) suggests the rescaling

$$\Phi(\kappa, z) = \frac{v(\eta)}{\mu^\alpha}\left\{\phi_N(z) + \mu\frac{C_0}{C_N}\eta q(z)\right\} \tag{B 6}$$

in terms of the 'inner' variable $\eta = (\kappa_N^{1/2} - \kappa)/\mu$. Combining then (B 6) with (3.8), taking into account (B 4*b*), it follows from (3.1)–(3.2) that, to leading order, $v(\eta)$ satisfies the (singular) integral equation

$$\eta v(\eta) - \tfrac{1}{2}(\alpha - 1)\int_{-\infty}^{\infty} \mathrm{d}\sigma \mathrm{e}^{-\pi\sigma/2}\operatorname{sech}\frac{\pi\sigma}{2}v(\eta - \sigma) = 0, \tag{B 7a}$$

subject to the condition

$$v(\eta) \sim \frac{C_N}{\eta^\alpha} \qquad (\eta \to \infty) \tag{B 7b}$$

that ensures matching with (B 3).

This linear problem can be readily solved via integral transform (Akylas & Yang 1995). Briefly, the solution is posed as

$$v(\eta) = \int_{\mathcal{L}} \mathrm{e}^{-\eta s}V(s)\mathrm{d}s, \tag{B 8}$$

where the contour $\mathcal{L}$ extends from $s = 0$ to ∞ with $\operatorname{Re}\eta s > 0$ and $\operatorname{Im} s < 0$. With this choice of $\mathcal{L}$, $v(\eta)$ is analytic in $\operatorname{Im}\eta > 0$ and, hence, does not contribute to the singularities of $\widehat{\varphi}(k, z)$ in $\operatorname{Im} k < 0$, consistent with the upstream condition (2.1) that the excited lee waves vanish in $x < 0$. Upon substitution of (B 8) into (B 7*a*), $V(s)$

satisfies the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}s} - (\alpha - 1)\frac{V}{\sin s} = 0,$$

and, in view of the matching condition (B 7*b*),

$$V(s) \sim \frac{C_N}{\Gamma(\alpha)} s^{\alpha-1} \qquad (s \to 0),$$

where $\Gamma(\alpha)$ denotes the gamma function. The appropriate solution is

$$V(s) = C_N \frac{2^{\alpha-1}}{\Gamma(\alpha)} (\tan \tfrac{1}{2}s)^{\alpha-1},$$

and, using (B 8), it is seen that $v(\eta)$ has a simple-pole singularity at $\eta = 0$:

$$v(\eta) \sim \frac{(-2\mathrm{i})^{\alpha-1}}{\Gamma(\alpha)} \frac{C_N}{\eta} \qquad (\eta \to 0). \tag{B 9}$$

Finally, combining (B 9) with (3.8) and (B6), we have established that $\widehat{\varphi}(k, z)$ has a simple-pole singularity at $k = k_N$ and, by very similar reasoning, the same is true at $k = -k_N$:

$$\widehat{\varphi} \sim \mp C_N \frac{2^\alpha}{\Gamma(\alpha)} \mu^{1-\alpha} \exp\left(\mp \mathrm{i}\tfrac{1}{2}\pi(\alpha-1)\right) \frac{\exp(-\frac{1}{2}\pi k_N)}{k \mp k_N} \phi_N(z) \qquad (k \to \pm k_N). \tag{B 10}$$

According to (B 10), the residues of the poles of $\widehat{\varphi}(k, z)$ at $k = \pm k_N$, and hence the lee-wave amplitude, depend on the constant C_N, which has to be determined numerically. For this purpose, the solution to the problem (B 1)–(B 2*b*) is posed as a power series,

$$\Phi(\kappa, z) = \sum_{m=1}^{\infty} b_m(z) |\kappa|^{m-1} \tag{B 11}$$

with $b_1 = -\frac{1}{2}Aq(z)$ according to (3.9), and, upon substitution into (B 1)–(B 2*b*), one has an infinite sequence of boundary-value problems, analogous to (A 4)–(A 5*b*), for $b_m(z)$ $(m \geqslant 2)$. The value of C_N is then found from integrating these problems numerically making use of the fact that, in view of (B 3), the projection of $b_m(z)$ on $\phi_N(z)$ behaves like

$$C_N \kappa_N^{-(\alpha+m-1)/2} \frac{\alpha(\alpha+1)\cdots(\alpha+m-2)}{(m-1)!}$$

as $m \to \infty$.

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